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J. Math. Anal. Appl. 313 (2006) 163–176

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

A discrete Fourier method for solving strongly coupled mixed hyperbolic problems

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Received 5 September 2003

Available online 11 July 2005

Submitted by M.D. Gunzburger

Abstract

This paper provides a discrete Fourier method for constructing stable numerical solutions of strongly coupled mixed hyperbolic problems. Using Crank–Nicholson scheme the exact solution of the discretized problem is found. Then the stability of the discrete solution is analyzed and illustrative examples are included.

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Keywords: Coupled hyperbolic problem; Numerical solution; Stability analysis

1. Introduction

Coupled hyperbolic partial differential systems arise in microwave heating processes [7,13], optics [4], cardiology [18] and soil flows [19], for instance. In this paper we use semi-implicit matrix difference scheme, particular by the Crank–Nicholson scheme to construct stable numerical solutions of mixed problems described by

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$$Au_{xx}(x, t) - u_{tt}(x, t) = 0, \quad 0 < x < 1, \quad t > 0, \quad (1)$$

$$u(0, t) = 0, \quad t > 0, \quad (2)$$

$$Bu(1, t) + Cu_x(1, t) = 0, \quad t > 0, \quad (3)$$

$$u(x, 0) = F(x), \quad 0 \leq x \leq 1, \quad (4)$$

$$u_t(x, 0) = V(x), \quad 0 \leq x \leq 1, \quad (5)$$

where A, B, C are matrices in $\mathbb{C}^{s \times s}$ and the unknown $u(x, t)$ as well as functions $F(x)$ and $V(x)$ take values in \mathbb{C}^s . We assume that

$$\alpha(A) = \min\{\operatorname{Re}(z); z \text{ eigenvalue of } A\} > 0 \quad (6)$$

and

$$C \text{ is invertible.} \quad (7)$$

Explicit difference schemes for solving coupled parabolic problems have been proposed in [3,9], and for the hyperbolic case in [11] assuming that matrix A is symmetric and all its eigenvalues are real and positive. In this paper we use Crank–Nicholson scheme which permits the construction of stable solutions of problem (1)–(5) without assuming that matrix A is symmetric. Throughout this paper, the set of all the eigenvalues of a matrix P in $\mathbb{C}^{s \times s}$ is denoted by $\sigma(P)$ and its 2-norm, denoted by $\|P\|$ is defined by [6, p. 56]

$$\|P\| = \sup_{z \neq 0} \frac{\|Pz\|_2}{\|z\|_2},$$

where for a vector z in \mathbb{C}^s , $\|z\|_2$ is the usual norm of z . We denote by S^\dagger the Moore–Penrose pseudoinverse of the matrix S in $\mathbb{C}^{s \times q}$. An account of properties and applications of the concept may be found in [2,14]. The kernel of S denoted by $\operatorname{Ker} S$, coincides with the image of $I - S^\dagger S$, denoted by $\operatorname{Im}(I - S^\dagger S)$. We say that a subspace E of \mathbb{C}^n is invariant by the matrix $A \in \mathbb{C}^{s \times s}$ if $A(E) \subset E$. The property $A(\operatorname{Ker} G) \subset \operatorname{Ker} G$ is equivalent to the condition $GA(I - G^\dagger G) = 0$, see [12].

If P is a matrix in $\mathbb{C}^{s \times s}$, $f(z)$ is an holomorphic function defined on an open set Ω of the complex plane and $\sigma(P)$ lies in Ω , the holomorphic matrix functional calculus defines $f(P)$ as a matrix that may be computed as a polynomial in P of degree smaller than the minimal polynomial of P , see [5, p. 567]. In particular, if $\operatorname{Re}(z) > 0$ for all $z \in \sigma(P)$, then $\sigma(P)$ lies in $D_0 = \mathbb{C} \setminus]-\infty, 0]$ and considering $f(z) = \log(z)$ the principal branch of the complex logarithm, holomorphic in D_0 [15, p. 76], then for $z \in D_0$, the function $\sqrt{z} = \exp(\frac{1}{2} \log z)$ is holomorphic and $\sqrt{P} = \exp(\frac{1}{2} \log P)$ is a square root of P . The real line is represented by \mathbb{R} . i denotes the imaginary unity $\sqrt{-1} = i$.

This paper is organized as follows. Section 2 deals with the discretization of problem (1)–(5) using Crank–Nicholson scheme and contains some preliminary results about matrix equations and difference systems. Section 3 addresses the study of existence conditions of the discrete boundary value problem resulting from the discretization of problem (1)–(3), as well as the explicit construction of such solutions. In Section 4, using a discrete Fourier method, one construct a numerical solution of the discrete mixed problem. Finally, in Section 5 we prove that solutions are stable in the fixed station sense with respect to the time and an example is included.

2. Discretization and preliminaries

Let us divide the domain $[0, 1] \times [0, +\infty[$ into equal rectangles of sides $\Delta x = h$, $\Delta t = k$ and introduce coordinates of a typical mesh point (mh, nk) with $U(m, n) = u(mh, nk)$. Using Crank–Nicholson difference scheme, one gets the approximations [16,17]:

$$u_t(mh, nk) \approx \frac{U(m, n+1) - U(m, n)}{k}, \quad (8)$$

$$u_x(mh, nk) \approx \frac{U(m, n) - U(m-1, n)}{h}, \quad (9)$$

$$u_{xx}(mh, nk) \approx \frac{1}{2} \left[\frac{U(m+1, n+1) - 2U(m, n+1) + U(m-1, n+1)}{h^2} \right] + \frac{1}{2} \left[\frac{U(m+1, n) - 2U(m, n) + U(m-1, n)}{h^2} \right], \quad (10)$$

$$u_{tt}(mh, nk) \approx \frac{U(m, n+1) - 2U(m, n) + U(m, n-1))}{k^2}. \quad (11)$$

Let M be a positive integer, $h = 1/M$, $r = k/h$, $1 \leq m \leq M-1$ and let us substitute expressions (8)–(11) into (1)–(5), obtaining the mixed difference problem

$$\begin{aligned} & \frac{r^2}{2} A [U(m+1, n+1) - 2U(m, n+1) + U(m-1, n+1) \\ & \quad + U(m+1, n) - 2U(m, n) + U(m-1, n)] \\ & = U(m, n+1) - 2U(m, n) + U(m, n-1), \quad 1 \leq m \leq M-1, \quad n > 0, \end{aligned} \quad (12)$$

$$U(0, n) = 0, \quad n > 0, \quad (13)$$

$$BU(M, n) + MC[U(M, n) - U(M-1, n)] = 0, \quad n > 0, \quad (14)$$

$$U(m, 0) = F(mh) = f(m), \quad 0 \leq m \leq M, \quad (15)$$

$$\frac{1}{k} [U(m, 1) - U(m, 0)] = V(mh) = v(m), \quad 0 \leq m \leq M. \quad (16)$$

If we seek nonzero solutions of the boundary value problem (12)–(14) of the form

$$U(m, n) = G(n)H(m), \quad G(n) \in \mathbb{C}^{s \times s}, \quad H(m) \in \mathbb{C}^s. \quad (17)$$

Substituting (17) in (12), one gets

$$\begin{aligned} & \frac{r^2 A}{2} [G(n+1) + G(n)] [H(m+1) - 2H(m) + H(m-1)] \\ & = [G(n+1) - 2G(n) + G(n-1)] H(m). \end{aligned} \quad (18)$$

Subtracting from both sides of (18) the expression $\frac{\rho A}{2} [G(n+1) + G(n)] H(m)$, where $\rho \in \mathbb{R}$, and arranging the resulting expression, one gets

$$\frac{r^2 A}{2} [G(n+1) + G(n)] \left[H(m+1) - \left(2 + \frac{\rho}{r^2} \right) H(m) + H(m-1) \right]$$

$$-\left[\left(I - \frac{\rho A}{2}\right)G(n+1) - \left(2I + \frac{\rho A}{2}\right)G(n) + G(n-1)\right]H(m) = 0, \\ 1 \leq m \leq M-1, \quad n > 0. \quad (19)$$

Equation (19) holds if $\{H(m)\}$ and $\{G(n)\}$ satisfy the difference equations

$$H(m+1) - \left(2 + \frac{\rho}{r^2}\right)H(m) + H(m-1) = 0, \quad 1 \leq m \leq M-1, \quad (20)$$

and

$$\left(I - \frac{\rho A}{2}\right)G(n+1) - \left(2I + \frac{\rho A}{2}\right)G(n) + G(n-1) = 0, \quad n > 0. \quad (21)$$

Let ρ be a given real number lying in the interval

$$-4r^2 < \rho < 0. \quad (22)$$

Then the general vector solution of (20) takes the form

$$H(m) = \cos(m\theta)D + \sin(m\theta)E, \quad D, E \in \mathbb{C}^s, \quad (23)$$

where

$$0 < \theta < \pi, \quad \cos \theta = \frac{2r^2 + \rho}{2r^2}, \quad \rho = -4r^2 \sin^2(\theta/2). \quad (24)$$

Under hypothesis (6) as $\rho < 0$, the matrix $I - \rho A/2$ is invertible and the algebraic matrix equation

$$\left(I - \frac{\rho A}{2}\right)Z^2 - \left(2I + \frac{\rho A}{2}\right)Z + I = 0,$$

associated to (21) is equivalent to

$$Z^2 - \left(I - \frac{\rho A}{2}\right)^{-1} \left(2I + \frac{\rho A}{2}\right)Z + \left(I - \frac{\rho A}{2}\right)^{-1} = 0. \quad (25)$$

By [8,10], if $\{Z_0, Z_1\}$ is a pair of solutions of (25) such that $Z_0 - Z_1$ is invertible, called a complete set of solutions of (25), then the general $\mathbb{C}^{s \times q}$ solution of Eq. (21) is given by

$$G(n) = Z_0^n P + Z_1^n Q, \quad n \geq 0, \quad (26)$$

where, for given initial conditions $G(0)$ and $G(1)$ in $\mathbb{C}^{s \times q}$, one gets

$$P = (Z_0 - Z_1)^{-1}[(I - Z_1)G(0) - G(1)], \\ Q = (Z_0 - Z_1)^{-1}[(Z_0 - I)G(0) - G(1)]. \quad (27)$$

Taking ρ small enough so that

$$|\rho| < \frac{16}{\alpha(A)}, \quad (28)$$

by (6) one gets that $\operatorname{Re}(1 + \rho a/16) > 0$ for all $a \in \sigma(A)$ and thus there exist $\sqrt{1 + \rho A/16}$ in the sense given in the introduction. Hence we can write

$$\begin{aligned}
& (I - \rho A/2)^{-2}(I + \rho A/4)^2 - (I - \rho A/2)^{-1} \\
&= (I - \rho A/2)^{-2}[(I + \rho A/4)^2 - (I - \rho A/2)] \\
&= (I - \rho A/2)^{-2}(\rho A + \rho^2 A^2/16) \\
&= (I - \rho A/2)^{-2}\rho A(I + \rho A/16).
\end{aligned} \tag{29}$$

Since the matrix coefficients of (25) are mutually commutative, a pair of solutions of (25) is given by

$$\begin{aligned}
Z &= (I - \rho A/2)^{-1}(I + \rho A/4) \\
&\quad \pm \sqrt{(I - \rho A/2)^{-2}(I + \rho A/4)^2 - (I - \rho A/2)^{-1}},
\end{aligned} \tag{30}$$

where, by (29) one gets

$$\begin{aligned}
Z_0 &= (I - \rho A/2)^{-1}[I + \rho A/4 + i|\rho|^{1/2}\sqrt{A}\sqrt{I + \rho A/16}], \\
Z_1 &= (I - \rho A/2)^{-1}[I + \rho A/4 - i|\rho|^{1/2}\sqrt{A}\sqrt{I + \rho A/16}],
\end{aligned} \tag{31}$$

or

$$\begin{aligned}
Z_0 &= (I - \rho A/2)^{-1}[I + \rho A/4 + \rho^{1/2}\sqrt{A}\sqrt{I + \rho A/16}], \\
Z_1 &= (I - \rho A/2)^{-1}[I + \rho A/4 - \rho^{1/2}\sqrt{A}\sqrt{I + \rho A/16}].
\end{aligned} \tag{32}$$

By (31), (32) and the properties of the holomorphic matrix functional calculus, see [5, Chapter VII], matrices Z_0 and Z_1 can be computed as polynomials in the matrix A , and by Cayley–Hamilton theorem, as polynomials in A of degree $p - 1$, being p the degree of the minimal polynomial of A . Thus

$$Z_0 = \sum_{s=0}^{p-1} a_s A^s, \quad Z_1 = \sum_{s=0}^{p-1} b_s A^s \tag{33}$$

for some numbers $a_s, b_s, 0 \leq s \leq p - 1$.

3. The boundary value problem

In this section we study the existence of nonzero solutions of the boundary value problem (12)–(14). Note that $\{U(m, n)\}$ defined by (17) satisfies (13) if $\{H(m)\}$ satisfies (20) and $H(0) = 0$. The solution set of (20) satisfying $H(0) = 0$, takes the form

$$H(m) = \sin(M\theta)E, \quad E \in \mathbb{C}^s, \quad 1 \leq m \leq M - 1. \tag{34}$$

By (17), (26), (33) and (34) the boundary condition (14) if vector E of (34) satisfies

$$\{B \sin(M\theta) + MC[\sin(M\theta) - \sin((M - 1)\theta)]\}G(n)E = 0, \quad n > 0, \tag{35}$$

or if vectors P, Q in \mathbb{C}^s satisfy

$$\begin{aligned}
& \{B \sin(M\theta) + MC[\sin(M\theta) - \sin((M - 1)\theta)]\}A^j(P, Q) = 0, \\
& 0 \leq j \leq p - 1.
\end{aligned} \tag{36}$$

In order to find nonzero solutions of (12)–(14), we must find vectors P, Q , nonsimultaneously zero satisfying (36). Hence, a necessary and sufficient condition to obtain solutions of the form (17) is

$$L(\theta) = B \sin(M\theta) + MC[\sin(M\theta) - \sin((M-1)\theta)] \quad \text{is singular.} \quad (37)$$

Since C is invertible and $L(\theta)$ is singular if and only if $\sin(M\theta) \neq 0$, condition (37) is equivalent to

$$C^{-1}B + \frac{M[\sin(M\theta) - \sin((M-1)\theta)]I}{\sin(M\theta)} \quad \text{is singular.} \quad (38)$$

Condition (38) holds if there exist

$$\mu \in \sigma(-C^{-1}B) \cap \mathbb{R} \quad (39)$$

satisfying

$$\frac{M[\sin(M\theta) - \sin((M-1)\theta)]}{\sin(M\theta)} = \mu, \quad (40)$$

or the equivalent equation

$$\cot(M\theta) = \frac{\cos \theta - (1 - \mu/M)}{\sin \theta}. \quad (41)$$

With respect to (41) by [3,9], it is known that:

- (i) If $\mu \leq 0$, then there exist a solution θ_ℓ of (41) in $J_\ell =]\frac{(\ell-1)\pi}{M}, \frac{\ell\pi}{M}[$ for each ℓ with $1 \leq \ell \leq M-1$. Furthermore, if B is singular, then $\mu = 0$ and $\theta_\ell = (\frac{2\ell-1}{2M-1})\pi$.
- (ii) If $0 < \mu < 1$, then there exist a solution θ_ℓ of (41) in J_ℓ , for $1 \leq \ell \leq M-1$.
- (iii) If $\mu \geq 1$, there exist a solution θ_ℓ of (41) in J_ℓ , for $2 \leq \ell \leq M-1$ and in $J_1 =]0, \frac{\pi}{M}[$ no solution exist.

Note that $\{H(m)\}$ satisfies the vector boundary value problem

$$\begin{aligned} -H(m+1) + 2H(m) - H(m-1) &= \frac{-\rho}{r^2} H(m), \quad 1 \leq m \leq M-1, \\ H(0) &= 0, \quad BH(M) + MC[H(M) - H(M-1)] = 0, \quad \rho \in \mathbb{R}, \end{aligned} \quad (42)$$

and under hypothesis (39) it has been transformed into the scalar Sturm–Liouville problem

$$\begin{aligned} -h(m+1) + 2h(m) - h(m-1) &= \frac{-\rho}{r^2} h(m), \quad 1 \leq m \leq M-1, \\ h(0) &= 0, \quad h(M) = \frac{M}{M-\mu} h(M-1), \quad \rho \in \mathbb{R}, \end{aligned} \quad (43)$$

where every eigenvalue of (43) is an eigenvalue of (42) and eigenfunctions $\{H(m)\}$ of (42) associated to the eigenvalue ρ of (42) have all its vector components as eigenfunctions of the scalar problem (43). Since by [1, Chapter 11] the problem (43) has exactly $M-1$ eigenpairs, with eigenfunctions $\{h_\ell(m)\}_{\ell=1}^{M-1}$ mutually orthogonal with respect to

the weight function $\{\omega(m)\}$, with $\omega(m) = 1$ for $1 \leq m \leq M-1$, in the case (iii) corresponding to $\mu \geq 1$, it is necessary to find an eigenvalue different from those lying in J_ℓ for $2 \leq \ell \leq M-1$.

Let us take $\mu = 1$ and consider (20) with $\rho = 0$, whose solution set satisfying $H(0) = 0$ is given by

$$H(m) = md, \quad (44)$$

and the corresponding equation (21) for $\rho = 0$, has the solution set

$$G(n) = D + nE, \quad n \geq 0, \quad D, E \in \mathbb{C}^s. \quad (45)$$

Substituting (44)–(45) in (17) and imposing condition (14) it follows that

$$(C^{-1}B + I)(D, E) = 0. \quad (46)$$

Let us introduce the matrices $G(\mu)$ and $\tilde{G}(\mu)$ defined by

$$G(\mu) = C^{-1}B + \mu I, \quad \tilde{G}(\mu) = \begin{pmatrix} G(\mu) \\ G(\mu)A \\ \vdots \\ G(\mu)A^{p-1} \end{pmatrix} \in \mathbb{C}^{sp \times s}, \quad (47)$$

where p is the degree of the minimal polynomial of A . By (38), (40) and (47), for the case $\mu < 1$, $\rho_\ell = -4r^2 \sin^2(\theta_\ell/2)$ is an eigenvalue of (42) if

$$\text{rank}(\tilde{G}(\mu)) < s. \quad (48)$$

For the case $\mu = 1$, apart from $\rho_\ell = -4r^2 \sin^2(\theta_\ell/2)$ for $2 \leq \ell \leq M-1$, $\rho = 0$ is an eigenvalue of (42) if $\text{rank}(G(1)) < s$, but this condition is satisfied if (48) holds for $\mu = 1$.

We consider now Eq. (20) for $\rho > 0$. The solution set of (20) in this case takes the form

$$H(m) = \omega_1^m D + \omega_2^m E, \quad D, E \in \mathbb{C}^s,$$

and those satisfying $H(0) = 0$, take the form

$$H(m) = (\omega_1^m - \omega_2^m)D, \quad D \in \mathbb{C}^s, \quad (49)$$

where

$$\omega_1 = R + \sqrt{R^2 - 1}, \quad \omega_2 = R - \sqrt{R^2 - 1}, \quad R = \frac{2r^2 + \rho}{2r^2}. \quad (50)$$

The solutions of (21) for $\rho > 0$ are given by (26) because in this case Z_0, Z_1 are well defined by (32). Hence, $U(m, n)$ defined by (17) has the form

$$U(m, n) = (Z_0^n P + Z_1^n Q)(\omega_1^m - \omega_2^m), \quad P, Q \in \mathbb{C}^s. \quad (51)$$

By imposing the condition (14) to $\{U(m, n)\}$, one gets

$$\begin{aligned} \{B(\omega_1^M - \omega_2^M) + MC[(\omega_1^M - \omega_2^M) - (\omega_1^{M-1} - \omega_2^{M-1})]\}A^j(P, Q) &= 0, \\ 0 \leq j \leq p-1. \end{aligned} \quad (52)$$

We are interested in finding one eigenvalue associated to $\rho > 0$ for the case $\mu > 1$. To this end, let us consider the function Φ defined by

$$\Phi(R) = 1 - \frac{\omega_1^{M-1} - \omega_2^{M-1}}{\omega_1^M - \omega_2^M}, \quad (53)$$

and note that by (52) one gets that there are matricial solutions of (12)–(14) of the form (17) if, according with the notation of (47), one satisfies that

$$G(M\Phi(R)) = C^{-1}B + M\Phi(R)I \quad \text{is singular.} \quad (54)$$

Note that $\Phi(R)$ is continuous in $]1, +\infty[$ with

$$\lim_{R \rightarrow 1^+} \Phi(R) = 1/M, \quad \lim_{R \rightarrow +\infty^-} \Phi(R) = 1 \quad (55)$$

and hence, if μ satisfies $1 < \mu < M$, the function $\Phi(R)M - \mu$ changes its signs in $]1, +\infty[$ and thus there exist $R_1 \in]1, +\infty[$ such that

$$M\Phi(R_1) = \mu. \quad (56)$$

The corresponding value of ρ , $\rho_1 = \rho(R_1)$ defined by (50)

$$\rho_1 = 2r^2(R_1 - 1), \quad (57)$$

is an eigenvalue of (43) if $G(\mu) = G(M\Phi(R_1))$ satisfies (48). Under (48) for the cases $\mu < 1$ and $\mu > 1$, the pair of vectors (P, Q) satisfying (36) and (52), respectively, can be obtained in the form

$$(P, Q) = (I - \tilde{G}(\mu)^\dagger \tilde{G}(\mu))(P_0, Q_0), \quad P_0, Q_0 \in \mathbb{C}^s, \quad (58)$$

and for the case $\mu = 1$, vectors D, E , satisfying (46) are given by

$$(D, E) = (I - G(1)^\dagger G(1))(D_0, E_0), \quad D_0, E_0 \in \mathbb{C}^s, \quad (59)$$

see theorem of [14, p. 24]. Let us introduce the functions $U_\ell(m, n, \mu)$ given by

$$U_1(m, n, \mu) = (Z_0^n P_1 + Z_1^n Q_1) \sin(m\theta_1), \quad \mu < 1, \quad (60)$$

$$U_1(m, n, 1) = m(D + nE), \quad \mu = 1, \quad (61)$$

$$U_1(m, n, \mu) = (Z_0^n P_1 + Z_1^n Q_1)(\omega_1^m - \omega_2^m), \quad \mu > 1, \quad (62)$$

$$U_\ell(m, n, \mu) = (Z_0^n P_\ell + Z_1^n Q_\ell) \sin(m\theta_\ell), \quad 2 \leq \ell \leq M-1, \quad \mu \in \mathbb{R}, \quad (63)$$

where (P_ℓ, Q_ℓ) take the form of (P, Q) in (58) and (D, E) of (59). Summarizing, the following result has been established:

Theorem 3.1. *Let A, B, C be matrices in $\mathbb{C}^{s \times s}$ where C is invertible and satisfy (6) and (39). Let $M > \mu$, $r = k/h > 0$ and let p the degree of the minimal polynomial of A . Let $G(\mu), \tilde{G}(\mu)$ be defined by (47), let Z_0, Z_1 be defined by (31)–(32) and assume condition (48). Then:*

- (i) *If $\mu < 1$, there are nonzero solutions of problem (12)–(14) of the form (17) given by (60) and (63) where θ_ℓ are solutions of (41) in $J_\ell =]\frac{(\ell-1)\pi}{M}, \frac{\ell\pi}{M}[$ for $1 \leq \ell \leq M-1$ and (P_ℓ, Q_ℓ) are given by the right-hand side of (58). If $\mu = 0$, then $\theta_\ell = (\frac{2\ell-1}{2M-1})\pi$, $1 \leq \ell \leq M-1$.*

- (ii) If $\mu = 1$, then there are nonzero solutions of (12)–(14) of the form (17) given by (61) and (63) where θ_ℓ are solutions of (41) in J_ℓ and (P_ℓ, Q_ℓ) are given by the right-hand side of (58) and D, E by the right-hand side of (59).
- (iii) If $\mu > 1$, and ω_1, ω_2 are defined by (50), the problem (12)–(14) admits nonzero solutions of the form (17), given by (62) and (63) where (P_ℓ, Q_ℓ) are given by the right-hand side of (58).

4. The mixed problem

Once we have obtained nonzero solutions of the boundary value problem (12)–(14), we construct solutions of the mixed problem (12)–(16) using superposition principle. Theorem 3.1 suggests to distinguish several cases:

Case 1. $\mu < 1$.

By superposition of the solutions $U_\ell(m, n, \mu)$ given in (60) and (63) and by imposing to the candidate solutions of the mixed problem, conditions (15) and (16) it follows that

$$U(m, n) = \sum_{\ell=1}^{M-1} (Z_0 P_\ell + Z_1 Q_\ell) \sin(m\theta_\ell), \quad (64)$$

vectors P_ℓ, Q_ℓ must verify

$$f(m) = \sum_{\ell=1}^{M-1} (P_\ell + Q_\ell) \sin(m\theta_\ell), \quad (65)$$

and

$$kv(m) + f(m) = \sum_{\ell=1}^{M-1} (Z_0 P_\ell + Z_1 Q_\ell) \sin(m\theta_\ell). \quad (66)$$

Working component by component in (65)–(66) and taking into account that $\{\sin(m\theta_\ell)\}_{m=1}^{M-1}$ is the eigenfunction set of the scalar problem (43), the Fourier series theory for discrete Sturm–Liouville systems, see [1, Chapter 11], yields componentwise that

$$P_\ell + Q_\ell = \frac{\sum_{s=1}^{M-1} f(s) \sin(s\theta_\ell)}{\sum_{s=1}^{M-1} \sin^2(s\theta_\ell)}, \quad (67)$$

$$Z_0 P_\ell + Z_1 Q_\ell = \frac{\sum_{s=1}^{M-1} [kv(s) + f(s)] \sin(s\theta_\ell)}{\sum_{s=1}^{M-1} \sin^2(s\theta_\ell)}. \quad (68)$$

By (27) or directly from (67)–(68) it follow that

$$P_\ell = (Z_0 - Z_1)^{-1} \frac{\sum_{s=1}^{M-1} [kv(s) - (Z_1 - I)f(s)] \sin(s\theta_\ell)}{\sum_{s=1}^{M-1} \sin^2(s\theta_\ell)}, \quad (69)$$

$$Q_\ell = (Z_0 - Z_1)^{-1} \frac{\sum_{s=1}^{M-1} [(Z_0 - I)f(s) - kv(s)] \sin(s\theta_\ell)}{\sum_{s=1}^{M-1} \sin^2(s\theta_\ell)}. \quad (70)$$

But in order to define a solution of the problem (12)–(16), vector P_ℓ , Q_ℓ must belong to $\text{Ker}(\tilde{G}(\mu))$, see Theorem 3.1. Taking into account that Z_0 , Z_1 and $(Z_0 - Z_1)^{-1}$ are polynomials in A of degree $p - 1$ at most, from the previous comments, the function $\{U(m, n)\}$ defined by (64), (69), (70) is a solution of (12)–(16) if

$$\{f(m), v(m); 1 \leq m \leq M - 1\} \subset \text{Ker } \tilde{G}(\mu), \quad (71)$$

or, in the equivalent form

$$\{f(m), v(m); 1 \leq m \leq M - 1\} \subset \text{Ker } G(\mu), \quad (72)$$

$\text{Ker } G(\mu)$ is an invariant subspace by A .

By the properties of the Moore–Penrose pseudoinverse, condition (72) can be expressed in the form, see [12],

$$G(\mu)A(I - G(\mu)^\dagger G(\mu)) = 0. \quad (73)$$

Case 2. $\mu = 1$.

In this case, the only difference with respect to the previous case is that there are only $M - 2$ eigenfunction of (43) of the form $\sin(m\theta_\ell)$ instead of $M - 1$, and in its place the eigenfunction

$$h_1(m) = m. \quad (74)$$

Since the set $\{h_\ell(m)\}_{\ell=1}^{M-1}$ with $h_\ell(m) = \sin(m\theta_\ell)$ for $2 \leq \ell \leq M - 1$ are mutually orthogonal with respect to the weight function $\omega(m)$ with $\omega(m) = 1$, $1 \leq m \leq M - 1$, we may proceed in this case as in the previous one, obtaining the solution

$$U(m, n) = m(D + nE) + \sum_{\ell=2}^{M-1} (Z_0^n P_\ell + Z_1^n Q_\ell) \sin(m\theta_\ell), \quad (75)$$

where

$$D = \frac{1}{(\frac{M^3}{3} - \frac{M^2}{2} + \frac{M}{6})} \sum_{s=1}^{M-1} sf(s), \quad (76)$$

$$E = \frac{k}{(\frac{M^3}{3} - \frac{M^2}{2} + \frac{M}{6})} \sum_{s=1}^{M-1} sv(s), \quad (77)$$

and for $2 \leq \ell \leq M - 1$, P_ℓ and Q_ℓ defined by (69) and (70), respectively, under hypothesis (72) with $\mu = 1$. Note that under this hypothesis $(P_\ell, Q_\ell) \in \text{Ker } \tilde{G}(1)$ and $(D, E) \in \text{Ker } G(1)$.

Case 3. $\mu > 1$.

This case is like case 2 with the replacement of $h_1(m) = m$ given in (74), by

$$h_1(m) = \omega_1^m - \omega_2^m, \quad (78)$$

where ω_1 and ω_2 are defined by (50). According with Theorem 3.1 and superposition principle we seek the solution of (12)–(16) of the form

$$U(m, n) = (Z_0^n P + Z_1^n Q)(\omega_1^m - \omega_2^m) + \sum_{\ell=2}^{M-1} (Z_0^n P_\ell + Z_1^n Q_\ell) \sin(m\theta_\ell). \quad (79)$$

In this case, again $\{h_\ell(m)\}_{\ell=1}^{M-1}$ defined by (78) and $h_\ell(m) = \sin(m\theta_\ell)$ with $2 \leq \ell \leq M-1$, define the orthogonal set of eigenfunction of (43). Using discrete Fourier series theory, one gets that

$$P = (Z_0 - Z_1)^{-1} \frac{\sum_{s=1}^{M-1} [kv(s) - (Z_1 - I)f(s)](\omega_1^s - \omega_2^s)}{\sum_{s=1}^{M-1} (\omega_1^s - \omega_2^s)^2}, \quad (80)$$

$$Q = (Z_0 - Z_1)^{-1} \frac{\sum_{s=1}^{M-1} [(Z_0 - I)f(s) - kv(s)](\omega_1^s - \omega_2^s)}{\sum_{s=1}^{M-1} (\omega_1^s - \omega_2^s)^2}, \quad (81)$$

where P_ℓ, Q_ℓ are defined in (69), (70), respectively, under hypothesis (72) with $\mu > 1$. Summarizing, the following result has been established:

Theorem 4.1. Assume the hypotheses and the notation of Theorem 3.1, together with (72). Then problem (12)–(16) admits a solution given by (64), (69), (70) if $\mu < 1$; by (75)–(77), if $\mu = 1$, and by (79), (69)–(70), (80)–(81) if $\mu > 1$.

5. Stability

In this section we study the stability of solutions $U(m, n)$ of problem (12)–(16), in the fixed station sense with respect to the time. What means that given $T > 0$ and M positive integer, both fixed, the sequence $U(m, n)$ remains bounded as $n \rightarrow \infty$, $k > 0$, but with the restriction $1 \leq n \leq N$, $Nk = T$, and for all $1 \leq m \leq M$.

By Theorems 3.1 and 4.1, it is clear that the stability of the solutions constructed by Theorem 4.1 is closely related to the behavior of $\{Z_0^n\}, \{Z_1^n\}$ as $n \rightarrow \infty$, $k \rightarrow 0$ and $Nk = T$. Note that in the notation of Section 2, $r = k/h_0$, $h_0 = 1/M$ is fixed, and $k \rightarrow 0$ if and only if $r \rightarrow 0$. Writing (31) in terms of k , taking into account that $\rho_\ell = -\frac{4}{h^2} \sin^2(\theta_\ell/2)k^2$, we have

$$Z_0 = Z_0(k) = \left(I + \frac{2}{h_0^2} \sin^2(\theta_\ell/2) Ak^2 \right)^{-1} \times \left(I - \frac{\sin^2(\theta_\ell/2)}{h_0^2} Ak^2 + \frac{i}{h_0} \sin(\theta_\ell/2) k \sqrt{A} \sqrt{I - \frac{\sin^2(\theta_\ell/2)}{4h_0^2} Ak^2} \right) \quad (82)$$

with θ_ℓ fixed. By changing the sign of the third term inside the bracket one gets the expression of $Z_1 = Z_1(k)$. For the sake of clarity in the presentation let us keep the notation

of (31) with $\rho = \rho_\ell = -\frac{4}{h^2} \sin^2(\theta_\ell/2)k^2$, and let us consider the holomorphic function in the disk $|z| < 16/|\rho_\ell|$, for fixed ρ_ℓ , defined by

$$h(z) = \sqrt{1 + \rho z/16}, \quad |z| < 16/|\rho|. \quad (83)$$

Regarding the Taylor series expansion of $h(z)$ and the image of $h(z)$ throughout the matrix functional calculus, acting of A , taking ρ small enough, so that $|\rho| < 16/\|A\|$, see [5], one gets

$$\|h(A)\| = \|\sqrt{I + \rho A/16}\| \leq 1 + O(|\rho|) = 1 + O(k), \quad \text{as } k \rightarrow 0. \quad (84)$$

Note also that by the perturbation lemma, [5], for $|\rho| < 2/\|A\|$ one gets

$$\|(I - \rho A/2)^{-1}\| \leq (1 - \|\rho A/2\|)^{-1} = O(1), \quad \text{as } k \rightarrow 0. \quad (85)$$

By (31) and (84), (85) it follows that

$$\|Z_0(k)\| \leq 1 + O(k), \quad \|Z_1(k)\| \leq 1 + O(k), \quad \text{as } k \rightarrow 0. \quad (86)$$

Note also that by (31), we have

$$\|Z_0(k) - Z_1(k)\| = O(k), \quad \|(Z_0(k) - Z_1(k))^{-1}\| = O(k^{-1}), \quad \text{as } k \rightarrow 0. \quad (87)$$

Finally, note that $Z_0 - I$ and $Z_1 - I$ can be written in the form

$$\begin{aligned} Z_0 - I &= Z_0 - (I - \rho A/2)^{-1}(I - \rho A/2) \\ &= (I - \rho A/2)^{-1}[(I + \rho A/4 + i|\rho|^{1/2}\sqrt{A}\sqrt{I + \rho A/16}) - (I - \rho A/2)] \\ &= (I - \rho A/2)^{-1}[3\rho A/4 + i|\rho|^{1/2}\sqrt{A}\sqrt{I + \rho A/16}], \end{aligned} \quad (88)$$

and

$$Z_1 - I = (I - \rho A/2)^{-1}[3\rho A/4 - i|\rho|^{1/2}\sqrt{A}\sqrt{I + \rho A/16}]. \quad (89)$$

By (84), (85), (88) and (89), one gets

$$\|Z_0 - I\| = O(k), \quad \|Z_1 - I\| = O(k), \quad \text{as } k \rightarrow 0. \quad (90)$$

Note that fixed $M > 0$, by (86), (87) and (90) the Fourier vector coefficients P_ℓ , Q_ℓ , given by (69)–(70), as well as P , Q given by (80)–(81) both satisfy

$$\begin{aligned} \|P_\ell\| &= O(1), \quad \|Q_\ell\| = O(1), \quad \|P\| = O(1), \quad \|Q\| = O(1), \\ &\text{as } k \rightarrow 0. \end{aligned} \quad (91)$$

Also if $Nk = T$ and $1 \leq n \leq N$ by (86), one gets

$$\|Z_0^n\| \leq (1 + O(k))^n \leq e^{nkL} \leq e^{NkL} = e^{TL}, \quad \|Z_1^n\| \leq e^{TL}, \quad (92)$$

for some constant L so that $\|Z_0\| \leq 1 + Lk$, $\|Z_1\| \leq 1 + Lk$.

Note also that for the case $\mu = 1$, the Fourier coefficients D , E defined by (76), (77) satisfy

$$\|D\| = O(1), \quad \|E\| = O(k), \quad \text{as } k \rightarrow 0. \quad (93)$$

Hence, the expression (61), for $1 \leq m \leq M-1$, $1 \leq n \leq N = T/k$, satisfies

$$\|m(D + nE)\| \leq M\|D\| + M\|E\|T/k = O(1), \quad \text{as } k \rightarrow 0, \quad (94)$$

because of (93). Summarizing, the following result has been established:

Theorem 5.1. Under hypothesis of Theorem 4.1, the solution $U(m, n)$ of problem (12)–(16) is stable with respect to the time in the fixed station sense, for all eigenvalue $\mu \in \sigma(-C^{-1}B) \cap \mathbb{R}$.

6. Example

Example 6.1. Consider problem (1)–(5) with matrices

$$A = \begin{pmatrix} -4 & 5 & 5/3 \\ -8 & 9 & 8/3 \\ -3 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -4 & 3/2 & -2 \\ -18 & 9/2 & -8/3 \\ -10 & 3/2 & -4/3 \end{pmatrix},$$

$$C = \begin{pmatrix} -2 & 1 & 3 \\ 2 & 3 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

and vector functions

$$f(m) = \left(\frac{5}{3}mh, \frac{8}{3}mh, mh \right)^T, \quad v(m) = \left(\frac{5}{3}e^{mh}, \frac{8}{3}e^{mh}, e^{mh} \right)^T.$$

Then C is invertible and

$$-C^{-1}B = \begin{pmatrix} 2 & 0 & 0 \\ 5 & -3/2 & 1 \\ 1 & 0 & 1/3 \end{pmatrix}, \quad \sigma(-C^{-1}B) = \{2, -3/2, 1/3\},$$

$$\sigma(A) = \{1, 5\}.$$

Taking $\mu = 2$, we have

$$G(2) = C^{-1}B + 2I = \begin{pmatrix} 0 & 0 & 0 \\ -5 & 7/2 & -1 \\ -1 & 0 & 5/3 \end{pmatrix}.$$

Furthermore, the minimal polynomial of matrix A is given by

$$m(\lambda) = (5 - \lambda)(\lambda - 1)^2$$

with $p = \text{degree}(m(\lambda)) = 3$. Hence

$$\tilde{G}(2) = \begin{pmatrix} 0 & 0 & 0 \\ -5 & 7/2 & -1 \\ -1 & 0 & 5/3 \\ 0 & 0 & 0 \\ -5 & 7/2 & -1 \\ -1 & 0 & 5/3 \\ 0 & 0 & 0 \\ -5 & 7/2 & -1 \\ -1 & 0 & 5/3 \end{pmatrix}, \quad \text{rank } \tilde{G}(2) = 2 < 3.$$

Thus hypothesis (48) holds true for $\mu = 2$. Furthermore,

$$G(2)^\dagger = \begin{pmatrix} 0 & -40/343 & -111/686 \\ 0 & 34/343 & -30/343 \\ 0 & -24/343 & 345/686 \end{pmatrix}$$

and $\text{Ker } G(2)$ is invariant subspace by A because $G(2)A[I - G(2)^\dagger G(2)] = 0$.

We also have

$$\{f(m), v(m); 1 \leq m \leq M-1\} \subset \text{Ker } G(2).$$

By Theorem 3.1, the solution given by (79) is stable in the fixed station sense with respect to the time.

It is important to point out that the eigenvalues $\mu = -3/2$ and $\mu = 1/3$ do not satisfy the hypothesis of Theorem 3.1 because $\text{rank } \tilde{G}(\mu) = 3$ in both cases $\mu = -3/2$ and $\mu = 1/3$.

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